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LECTURE NOTES

<u>“KINEMATICS”</u>

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MECHANICS

Chapter 1. KINEMATICS

I. KINEMATICS OF THE TRANSLATIONAL MOTION

1. The basic concepts and definitions

The general study of the relationships between motion, forces and energy is called *mechanics*. It is a large field and its study is essential to understanding physics, therefore, these chapters appear first. Mechanics can be divided into subdisciplines by combining and recombining its different aspects. Three of them are given special names. The study of motion without regard to the forces or energies that may be involved is called *kinematics*. It is the simplest branch of mechanics. In contrast, the study of forces without changes in motion or energy is called *statics*. Lastly, the branch of mechanics that deals with both motion and forces together is called *dynamics*.

The models used in mechanics are: a material point (object which sizes can be neglected in comparison with distances on which this object moves), a system of material points, a rigid body, and continuous medium.

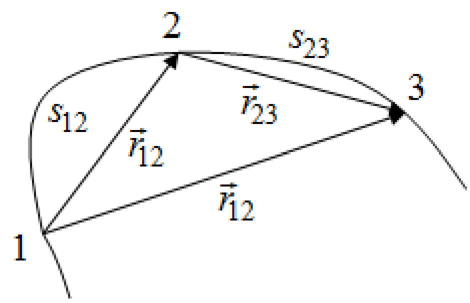
Motion happens in space and in time. According to Newton, «the space is a receptacle of all things, identical and motionless», «time flows in itself in regular intervals and irrespectively». According to Einstein, the space and time are inseparably linked with each other forming uniform four-dimensional «space-time».

There are several *types of motion*:

1. *Translational* motion results in a change of location (any straight line rigidly fastened with a body moves in a parallel way to itself);
2. *Rotational* motion occurs when an object spins (any straight line rigidly fastened with a body turns by an angle);
3. *Oscillatory* motion is repetitive and fluctuates between two locations;
4. *Chaotic* motion is predictable in theory but unpredictable in practice, which makes it appear random.

To describe the motion it is necessary to introduce a *frame of reference* consisting of a coordinate system and a clock.

2. Actual path. Distance. Displacement



An *actual path* (or *trajectory*) is a line which is described by a material point or an object during its three-dimensional motion. According to the form of a path the motion may be *rectilinear* or *curvilinear* motion. The special case of curvilinear motion is the rotation.

A *distance* is a scalar measure of interval between two locations measured along the actual path connecting them. The symbol for distance is Δs. The origin of this symbol is from the Latin word *spatium.* The distance is measured in meters, expresses by positive number and added arithmetically. The SI unit of distance is the meter.

$[s] = \text{meter} = \text{m}$

$$s_{13} = s_{12} + s_{23}.$$

A *displacement* is a vector measure of interval between two locations measured along the shortest path connecting them. $\Delta\vec{r}$ is the symbol for displacement. The origin of this symbol is from the Latin word *radius.* As a vector displacement has magnitude, direction and a point of origin. A displacement is always a straight line segment from one point to another point, even though the actual path of the moving object between the two points is curved. Furthermore, displacements are vector quantities and can be combined like other vector quantities.

$$\vec{r}_{13} = \vec{r}_{12} + \vec{r}_{23}.$$

The magnitude of displacement approaches distance as distance approaches zero.

The SI unit for displacement is the meter.

$[r] = \text{meter} = \text{m}.$

To specify the position of an object the concept of the position vector is to be introduced. The *position vector* $\vec{r}$ is defined as a vector that starts at the (user defined) origin and ends at the current position of the object. In general, the position vector $\vec{r}$ will be time dependent $\vec{r}(t)$.

$$\vec{r}(t) = x(t)\cdot\vec{e}_x + y(t)\cdot\vec{e}_y + z(t)\cdot\vec{e}_z,$$

where $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are unit vectors of the Cartesian frame.

3. Speed and velocity

Speed is the rate of change of distance with time. Speed is directly proportional to distance when time is constant and inversely proportional to time when distance is constant.

$$v = \frac{s}{t}.$$

If a particle moving in a straight line travels equal distances in equal periods of time, no matter how small these distances may be, the particle is said to be moving with *uniform speed*.

$$v = \frac{\Delta s}{\Delta t} = \frac{s}{t}.$$

In case of *non-uniform motion* the *mean* (or *average*) *speed* is introduced as total distance divided by total time, or, the distance traveled along a path divided by the time it takes to travel this distance:

$$\langle v\rangle = \frac{s}{t} = \frac{distance}{time}.$$

Car's speedometer shows its instantaneous speed, that is, the speed determined over a very small interval of time - an instant. Ideally this interval should be as close to zero as possible, but in reality we are limited by the sensitivity of our measuring devices. Mentally, however, it is possible to imagine calculating average speed over ever smaller time intervals until we have effectively calculated *instantaneous speed*. This idea is written symbolically as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

Speed is the first derivative of distance with respect to time.

Both instantaneous speed and average speed are completely described in terms of magnitude alone. Hence speed is scalar.

When both speed and direction are specified for the motion, the term velocity is used. *Velocity* is the rate of change of displacement with time. If, at any point of travel, $\Delta\vec{r}$ is the small change in displacement in a small period of time Δt, the velocity is given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}.$$

The *instantaneous velocity* is a velocity at that instant of time; it is the first derivative of displacement with respect to time.

Speed and velocity are related in much the same way as distance and displacement are related. Displacement is measured along the shortest path between two points and thus its magnitude is always less than or equal to the distance. The magnitude of the displacement approaches the distance as distance approaches zero. That is, distance and displacement are effectively the same (have the same magnitude) when the interval examined is "small". Since speed is based on distance and velocity is based on displacement, these two quantities are effectively the same (have the same magnitude) when the time interval examined is "small" or, in the language of calculus:

$$v = |\vec{v}| = \left|\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}\right| = \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t} \cdot \frac{\Delta s}{\Delta s} = \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta s} \cdot \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

The magnitude of an object's velocity approaches its speed as the time interval approaches zero.

The velocity vector can be decomposed into its three components. The components of velocity can be calculated using the components of position vector as

$$\vec{v} = \dot{\vec{r}} = v_x \cdot \vec{e}_x + v_y \cdot \vec{e}_y + v_z \cdot \vec{e}_z = \dot{x} \cdot \vec{e}_x + \dot{y} \cdot \vec{e}_y + \dot{z} \cdot \vec{e}_z = v \cdot \vec{e}_v,$$

where v_x, v_y, v_z are the components of velocity and $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are unit vectors of the Cartesian frame.

The SI unit of speed and velocity is the meter per second.

$[v] = \text{meter/second} = \text{m/s}$.

The fact that velocities are described as vectors means that they take on a central property of vectors, namely the *principle of superposition*. This principle states that the resultant velocity of an object is simply the vector sum of the velocities due to all things acting on the object.

$$\vec{v} = \vec{v}_1 + \vec{v}_2 .$$

4. Acceleration

Acceleration is defined as the rate of change of velocity with time

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}} .$$

Acceleration is the first derivative of a velocity with respect to time and the second derivative of a displacement with respect to time.

The SI unit of acceleration is the meter per second squared.

$$[a] = \mathrm{m/s^2} .$$

Any change in the velocity of an object results in an acceleration: increasing speed (what people usually mean when they say acceleration), decreasing speed (also called deceleration or retardation), or changing direction. Yes, that's right, a change in the direction of motion results in an acceleration even if the speed doesn't change. That's because acceleration depends on a change in velocity and velocity is a vector quantity – one with both magnitude and direction.

Taking into account that $\vec{v} = v \cdot \vec{e}_v$, we‘ll consider three types of motions:

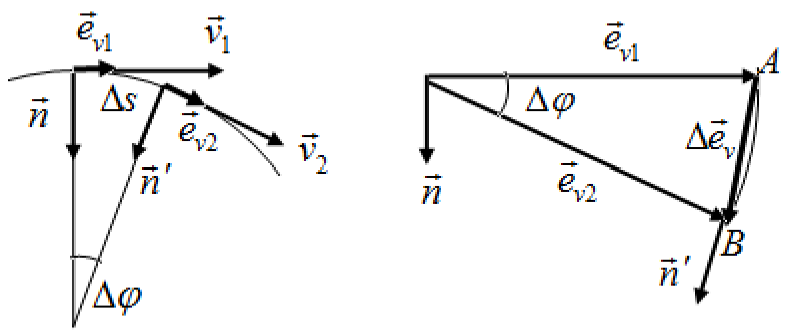
1. $v \neq \mathrm{const}$; $\vec{e}_v = \mathrm{const}$ (*non-uniform rectilinear motion*)

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot e_v) = \frac{dv}{dt} \cdot \vec{e}_v = \dot{v} \cdot \vec{e}_v = \vec{a}_\tau .$$

The *tangential acceleration* $\vec{a}_\tau$ is the component of acceleration vector tangent to the curvilinear path. It characterizes the change of velocity magnitude (speed).

2. $v = const$; $\vec{e}_v \neq const$ (*uniform curvilinear motion*)

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot e_v) = v \cdot \frac{d\vec{e}_v}{dt} = v \cdot \dot{\vec{e}}_v .$$



Let's find out what $\dot{\vec{e}}_v$ is. According to definition of a derivative $\dot{\vec{e}}_v = \frac{d\vec{e}_v}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{e}_v}{\Delta t}$. During the small period of time Δt the unit vector of a velocity

$\vec{e}_v$ turns through an angle of $\Delta\varphi = \dfrac{\Delta s}{R} = \dfrac{v \cdot \Delta t}{R}$.

When $\Delta t \to 0$ and $\Delta\varphi \to 0$, the chord $AB \cong AB$ (the arc length). Then $\Delta\vec{e}_v \approx \Delta\varphi \cdot \vec{n}'$. When Δt tends to zero, $\vec{n}' \to \vec{n}$.

$$\dot{\vec{e}}_v = \frac{d\vec{e}_v}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{e}_v}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\varphi \cdot \vec{n}'}{\Delta t} = \lim_{\Delta t \to 0} \frac{v \cdot \Delta t \cdot \vec{n}'}{R \cdot \Delta t} = \frac{v}{R} \cdot \vec{n}.$$

Therefore,

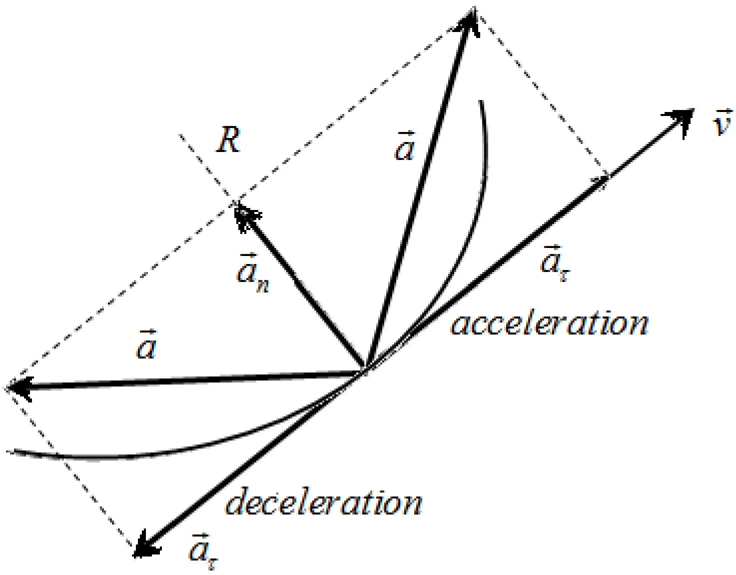
$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot \vec{e}_v) = v \cdot \frac{d\vec{e}_v}{dt} = v \cdot \dot{\vec{e}}_v = \frac{v^2}{R} \cdot \vec{n} = \vec{a}_n.$$

The *normal* (or *centripetal*, or radial) acceleration $\vec{a}_n$ is the component of the acceleration along the in-out direction, i.e. parallel to the radius. It characterizes the change of velocity direction.

3. $v \neq \text{const}$; $\vec{e}_v \neq \text{const}$ (*non-uniform curvilinear motion*)

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot e_v) = \frac{dv}{dt} \cdot \vec{e}_v + v \cdot \frac{d\vec{e}_v}{dt} = \vec{a}_\tau + \vec{a}_n,$$

$$a = |\vec{a}| = \sqrt{a_\tau^2 + a_n^2}.$$

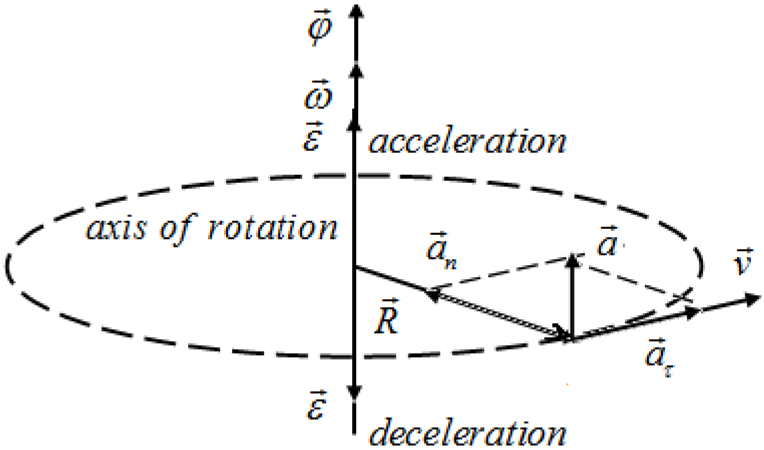


II. ROTATIONAL KINEMATICS

Rotational motion is the motion during which all points of a body move along the circular paths which centers lie on one straight line called an *axis of rotation*. Note the difference between *circular* motion and *rotational* motion. During circular motion, the axis of the motion is *outside* the object. During rotational motion, the axis of the motion is *inside* the moving object. A spinning wheel is in rotational motion; an object on the rim of the wheel describes circular motion.

1. The quantities describing the rotation

For uniform linear motion, a velocity is defined as the time rate of displacement. Similarly, for uniform rotary motion, an *angular velocity* is defined



as the time rate of angular displacement. An *angular displacement* $\vec{\varphi}$ is the angle about the axis of rotation through which the object turns. The direction of the vector $\vec{\varphi}$ is determined by *right hand rule*. Orient your right hand so that your thumb is perpendicular to the plane of rotation. If you can curl your fingers in the direction of rotation, your thumb will point in the direction of $\vec{\varphi}$.

An angular velocity is

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\varphi}}{\Delta t} = \frac{d\vec{\varphi}}{dt} = \dot{\vec{\varphi}}.$$

$$[\varphi] = \text{radian}, \quad [\omega] = \frac{\text{radian}}{\text{second}} = \text{s}^{-1}.$$

The change of an angular velocity is characterized by an *angular acceleration* $\vec{\varepsilon}$

$$\vec{\varepsilon} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} = \dot{\vec{\omega}}.$$

$$[\varepsilon] = \frac{\text{radian}}{\text{s}^2} = \text{s}^{-2}.$$

When the velocity increases (acceleration), vectors $\vec{\omega}$ and $\vec{\varepsilon}$ coincide in direction; when the velocity decreases (deceleration), they are opposite directed vectors.

Vectors $\vec{\varphi}$, $\vec{\omega}$ and $\vec{\varepsilon}$ are so-called *pseudo-vectors*, i.e., vectors which direction gets out conditionally.

2. Relationships between the parameters describing translational and rotational motions

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \varphi \cdot R}{\Delta t} = R \cdot \lim_{\Delta t \to 0} \frac{\Delta \varphi}{\Delta t} = R \cdot \omega,$$

$$\vec{v} = \left[\vec{\omega}, \vec{R} \right],$$

$$a_n = \frac{v^2}{R} = \omega^2 \cdot R,$$

$$v = \omega \cdot R \Rightarrow \dot{v} = \dot{\omega} \cdot R = \varepsilon \cdot R = a_\tau,$$

$$\vec{a}_\tau = \left[\vec{\varepsilon}, \vec{R} \right].$$